ST2351 Probability and Theoretical Statistics

Sample Exam Questions

To successfully pass the exam you need to answer any 2 out of 4 questions

1. (a) Define the conditional probability of an event $A$ given an event $B$.

(b) State and prove Bayes’ law. Any other law of probability may be used in the proof, as long as it is carefully stated.

(c) In the case *State of Connecticut v. Teal*, discrimination against black employees was alleged in a company.

   The company used a test to determine eligibility for promotion. Three hundred and seven people took the test, of which 48 were black ($B$) and 259 were white ($W$). Twenty six of the 226 that passed ($P$) were black, and 22 of the remaining 81 that failed ($F$) were black.

   Thus:
   
   $P(P) = \frac{226}{307}$, $P(F) = \frac{81}{307}$, $P(B|P) = \frac{26}{226}$ and $P(B|F) = \frac{22}{81}$.

   To discover if there was discrimination, the court in the case ruled that $P(P|B)$ had to be compared with $P(P|W)$. Use Bayes’ law to calculate these two probabilities.

(d) $X$ and $Y$ are continuous random variables. Define the conditional probability density function of $X$ given $Y$.

(e) Let $X$ and $Y$ have the following joint pdf:

   $$f_{X,Y}(x, y) = \begin{cases} 
   c(x + 2xy + y^2)^2, & \text{if } 0 < x < y < 1, \\
   0, & \text{otherwise},
   \end{cases}$$

   where $c$ is a constant.

   i. Calculate $c$ so that $f_{X,Y}$ is a legitimate joint density function.

   ii. Compute the conditional density function of $X$ given $Y$.

2. (a) Define a continuous random variable.

(b) Let $X$ be the time to failure of a certain engine component, which is exponentially distributed with a mean of 4 years.

   i. Write down the probability density function of $X$.

   ii. Calculate $P(2 \leq X \leq 5)$.

   iii. Give reasons why this probability $P(2 \leq X \leq 5)$ is actually the same as $P(2 < X \leq 5)$ and $P(2 < X < 5)$.

(c) Let $Y$ be uniformly distributed on the interval $[-1, 1]$.

   i. Compute the mean and variance of $Y$

   ii. What is the moment generating function of $Y$?

   iii. Let $Z$ be independent of $Y$ and also be uniformly distributed on the interval $[-1, 1]$.

   Show, by means of moment generating functions or otherwise, that the probability density function of $S = Y + Z$ is

   $$f_S(s) = \begin{cases} 
   0.5 + 0.25s, & \text{if } -2 \leq s \leq 0, \\
   0.5 - 0.25s, & \text{if } 0 < s \leq 2, \\
   0, & \text{otherwise}.
   \end{cases}$$

   You may use any theorem of moment generating functions, as long as it is carefully stated. You may use the fact that $\int se^{ts} ds = (e^{as} - 1)e^{ts}/t^2$.

3. (a) Define the joint probability mass function $p_{X,Y}(x,y)$ for two discrete random variables $X$ and $Y$. 

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(b) Define the expected value of \( g(X, Y) \), where \( g \) is a real-valued function, and \( X \) and \( Y \) are discrete random variables with joint mass function \( p_{X,Y}(x, y) \). Show that expectation is linear, that is \( E(aX + bY) = aE(X) + bE(Y) \) for any two constants \( a \) and \( b \).

(c) \( X \) and \( Y \) have joint mass function given in the following table:

<table>
<thead>
<tr>
<th>( X )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>( c )</td>
<td>2c</td>
<td>0</td>
<td>5c</td>
</tr>
<tr>
<td>1</td>
<td>( c )</td>
<td>( c )</td>
<td>3c</td>
<td>2c</td>
</tr>
<tr>
<td>2</td>
<td>6c</td>
<td>2c</td>
<td>2c</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>( c )</td>
<td>3c</td>
<td>3c</td>
<td>( c )</td>
</tr>
</tbody>
</table>

where \( c \) is a constant

i. Explain why it must be that \( c = 1/33 \).
ii. Compute the marginal mass functions of \( X \) and \( Y \).
iii. Compute the covariance of \( X \) and \( Y \).
iv. Are \( X \) and \( Y \) independent?

4. (a) \( X \) and \( Y \) are jointly continuous random variables with joint probability density function:

\[
f_{X,Y}(x, y) = \begin{cases} 
4xy, & 0 \leq x \leq 1, 0 \leq y \leq 1, \\
0, & \text{otherwise}.
\end{cases}
\]

Determine the joint probability density function of the random variables \( U = X + Y \) and \( V = X - Y \). You may use any theorem without proof as long as it is quoted carefully.

(b) State the central limit theorem.

(c) A point moves on the real line in a series of randomly sized jumps. The size of the \( i \)th jump is denoted \( X_i \). The \( X_i \) are independent and identically distributed uniform random variables on the interval \([-1, 1]\). Let \( S_n \) be the distance of the point from its starting position after \( n \) moves. Find an approximate value for \( P(-120 < S_{10800} < 60) \). The mean and variance of the \( X_i \) are 0 and 1/3 respectively.